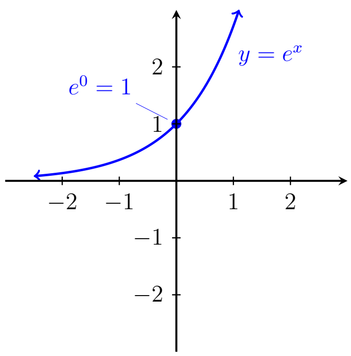
Objectives:

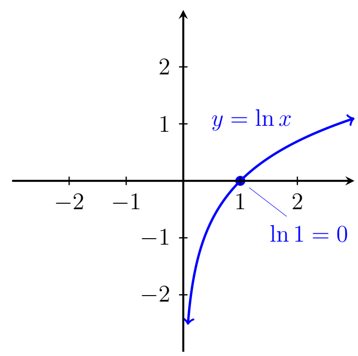
- Find derivatives of logarithmic functions.

Review of logarithmic functions:

$y = e^x$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$



Inverse: $y = \ln(x)$
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$



Important facts:

$$y = e^x \text{ means } \ln y = x \qquad \text{and} \qquad y = a^x \text{ means } \log_a y = x$$

$\log_2 8 = 3$	is equivalent to	$2^3 = 8$
$\log_{10} 100 = 2$	is equivalent to	$10^2 = 100$
$\log_{10} 0.001 = -3$	is equivalent to	$10^{-3} = \dfrac{1}{1000} = 0.001$
$\ln \sqrt{e} = \dfrac{1}{2}$	is equivalent to	$e^{1/2} = \sqrt{e}$
$y = \log_a x$	is equivalent to	$a^y = x$
$\ln y = x$	is equivalent to	$y = e^x$

Solving Equations:

Solve $5e^{0.34t} = 6$ for t.

$$\begin{aligned} e^{0.34t} &= \frac{6}{5} \\ \ln\left(e^{0.34t}\right) &= \ln\left(\frac{6}{5}\right) \\ 0.34t &= \ln\frac{6}{5} \\ t &= \frac{\ln\frac{6}{5}}{0.34} \end{aligned}$$

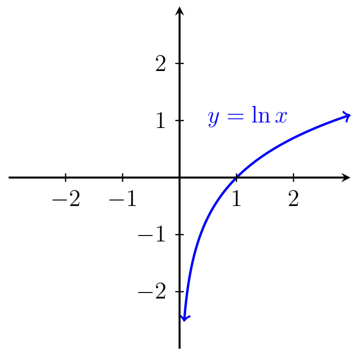
Solve $\log_2(x) + \log_2(x-1) = 1$.
(Hint: use laws of logs)

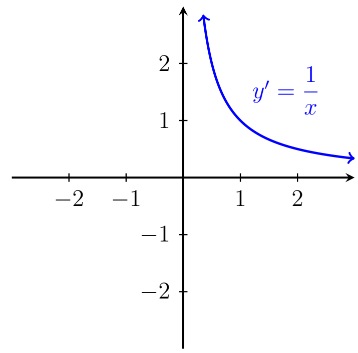
$$\begin{aligned} \log_2\big(x(x-1)\big) &= 1 \\ x(x-1) &= 2^1 \\ x^2 - x &= 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$

So $x = 2$ or $x = -1$ but substituting shows $x = -1$ is not in the domain. So $x = 2$ is the solution.

Derivatives of logarithmic functions:

1. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
2. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{\ln a} \cdot \dfrac{1}{x}$





Proof: (of 1.)

Step 1: $y = \ln x$

Step 2: $e^y = x$ (use inverses)

Step 3: $e^y \cdot y' = 1$ (differentiate)

Step 4: $y' = \dfrac{1}{e^y}$ (solve for y')

Step 5: $y' = \dfrac{1}{x}$ (substitute to write in terms of x)

Try the proof for 2 at home!

Examples: Find the derivatives of the following functions using our new formulas.

1. $f(x) = x\ln(x)$

$$f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

2. $f(x) = \ln\big(\arctan(x)\big)$

$$f'(x) = \frac{1}{\arctan x} \cdot \frac{1}{1+x^2}$$

3. $f(x) = \sqrt{\log_2(x)} = (\log_2 x)^{1/2}$

$$f'(x) = \frac{1}{2}(\log_2 x)^{-1/2} \cdot \frac{1}{x\ln 2}$$

4. $f(x) = \arcsin\left(e^{\tan(x^2)}\right)$

$$f'(x) = \frac{1}{\sqrt{1-e^{2\tan(x^2)}}} \cdot e^{\tan(x^2)} \cdot \sec^2(x^2) \cdot 2x$$

5. $f(x) = \ln\left(\dfrac{x^2\sqrt{x-1}}{(x+3)^4}\right) = \ln(x^2) + \ln\left(\sqrt{x-1}\right) - \ln\left((x+3)^4\right) = 2\ln x + \dfrac{1}{2}\ln(x-1) - 4\ln(x+3)$

$$f'(x) = \frac{2}{x} + \frac{1}{2(x-1)} - \frac{4}{x+3}$$

Logarithmic Differentiation: Why would we take logarithms to take derivatives?

- Use it with functions that have products/quotients/powers, like $f(x) = \dfrac{x^4\sqrt{x^2+1}}{(3x+4)^2}$
- Use if if there is a variable in the base <u>and</u> the exponent, like $f(x) = x^{\sin x}$, since neither the power rule nor the exponential rule apply.

Examples Find $f'(x)$ for the following functions.

1. $f(x) = \dfrac{x^4\sqrt{x^2+1}}{(3x+4)^2}$.

Step 1: Use y for $f(x)$

$$y = \frac{x^4\sqrt{x^2+1}}{(3x+4)^2}$$

Step 2: Take ln of both sides

$$\ln y = \ln\left(\frac{x^4\sqrt{x^2+1}}{(3x+4)^2}\right)$$

Step 3: Use laws of logs to simplify

$$\ln y = 4\ln x + \frac{1}{2}\ln(x^2+1) - 2\ln(3x+4)$$

Step 4: Differentiate implicitly

$$\frac{1}{y}\cdot y' = \frac{4}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{2}{3x+4}\cdot 3$$

Step 5: Solve for y'

$$y' = y\left(\frac{4}{x} + \frac{x}{x^2+1} - \frac{6}{3x+4}\right) = \left(\frac{x^4\sqrt{x^2+1}}{(3x+4)^2}\right)\left(\frac{4}{x} + \frac{x}{x^2+1} - \frac{6}{3x+4}\right)$$

2. $f(x) = x^{\sin x}$

$$\begin{aligned} y &= x^{\sin x} \\ \ln y &= \ln\left(x^{\sin(x)}\right) = \sin x \cdot \ln x \\ \frac{1}{y}y' &= \frac{\sin(x)}{x} + \ln x\cos x \\ y' &= y\left(\frac{\sin(x)}{x} + \ln x\cos x\right) \\ &= x^{\sin x}\left(\frac{\sin(x)}{x} + \ln x\cos x\right) \end{aligned}$$